ON THE ANALYSIS OF LINE-LOADED CYLINDRICAL SHELLS

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Apparently Odquist [1] first investigated the state of stress of a circular cylindrical shell, loaded along lines on a surface, in 1946. A shell reinforced at the endfaces by rigid bulkheads and loaded along a segment of the generatrix by a uniformly distributed radial and circumferential loading as well as a circumferential bending moment was considered here. The solution was constructed by starting from an approximate equation obtained under the assumption that there were no circumferential deformations and shears of the middle surface, and also no longitudinal bending moments and torques. At the same time the Shoessow and Kooistra experiments [2] (1945) showed that there is no foundation for not taking account of the longitudinal moment since the ratio between the longitudinal and circumferential bending moments may reach 0.5. A freely supported shell, loaded along the generator by radial forces and circumferential moments, was considered in [3] in 1954 by utilizing shallow shell equations. The solution was constructed by starting from a system of homogeneous equations for a shell slit along the loading line. The external forces and the desired quantities were expanded in Fourier series in the longitudinal coordinate. Such a method of solution is not efficient if the series for the external forces converges slowly.

In considering herein a closed circular cylindrical shell loaded along an arbitrary line of the middle surface, the method of computation is based on using the Green's function, the solution due to a concentrated force. Hence, a particular integral is constructed which corresponds to the solution of the problem for an infinite shell. The boundary conditions on the endfaces may be satisfied by adding on the solution of the homogeneous equations.

1. It is known [4] that the solution of the system of equilibrium equations in displacements is equivalent to the solution of three separate linear differential equations in the three displacement functions Ψ_j:

$$D\Psi_j + fX_j = 0 \qquad (j = 1, 2, 3) \tag{1.1}$$

where D is an eighth order linear partial differential operator; f is a known constant; X_j are components of the external surface loading intensity directed, respectively, along the generator, the arc of a circle, and along the circumference and normal to the shell surface.

In considering closed cylindrical shells the functions X_j in (1.1) are periodic in the circumferential direction. Let a linear force q with the components q_j ($j = 1, 2, 3$) act along some arbitrary line $l(\xi_0, \theta_0)$ of the shell surface. In this case the X_j in (1.1) may be represented as

$$X_j(\xi, \theta) = \frac{1}{\pi^2 R}\int_l q_j(\xi_0, \theta_0)\, \delta(\xi - \xi_0, \theta - \theta_0)\, dl \qquad \left(\xi = \frac{x}{R}, \ \theta = \frac{y}{R}\right)$$

$$dl = \sqrt{d\xi^2 + d\theta^2} \tag{1.2}$$

Here x, y are the longitudinal and circumferential coordinates of the surface; R is the radius of the shell middle surface; $\delta(\xi, \theta)$ a periodic delta-function which may be represented in the form

$$\delta(\xi, \theta) = \left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta\right) \int_0^{\infty} \cos \mu\xi \, d\mu \qquad (1.3)$$

We represent the solution of (1.1) by integrals analogous to (1.2):

$$\Psi_j(\xi, \theta) = \frac{f}{\pi^2 R} \int_l q_j(\xi_0, \theta_0) \, G(\xi - \xi_0, \theta - \theta_0) \, dl \qquad (1.4)$$

For an arbitrary external loading q_j the problem reduces to finding a Green's function G independent of the form of q_j. We have the following Eq. for G:

$$DG(\xi, \theta) + \delta(\xi, \theta) = 0 \qquad (1.5)$$

The solution of (1.5) is known. Juan [5] first obtained it in 1946 in considering the effect of a concentrated radial force on a shallow cylindrical shell. In 1951 Darevskii [6] generalized the solution of (1.5) for the shallow shell equations in the Love version.

Knowing the solution (1.4) due to a normal linear force q, we can obtain the solution due to the linear bending moments M_1 and M_2 acting along the line l, if we understand these moments to be given by the limiting relationships

$$M_1 = R \lim_{\Delta\xi \to 0} (2\Delta\xi q_3), \qquad M_2 = R \lim_{\Delta\theta \to 0} (2\Delta\theta q_3)$$

Such a solution is

$$\Psi_{3M}(\xi, \theta) = \frac{f}{\pi^2 R^2} \int_l \left[M_1(\xi_0, \theta_0) \frac{\partial}{\partial \xi} + M_2(\xi_0, \theta_0) \frac{\partial}{\partial \theta}\right] G(\xi - \xi_0, \theta - \theta_0) \, dl \qquad (1.6)$$

If the functions (1.4) have been found, then the axial u, the circumferential v, and the radial w displacements of the middle surface are determined by means of Formulas

$$u = \sum_{j=1}^{3} D_{j1} \Psi_j, \qquad v = \sum_{j=1}^{3} D_{j2} \Psi_j, \qquad w = \sum_{j=1}^{3} D_{j3} \Psi_j \qquad (1.7)$$

where $D_{ji} = D_{ij}$ are linear differential operators. For the shell theory version presented in the monograph [4] we have:

$$D_{11} = \frac{1-\nu}{2} \frac{\partial}{\partial \xi^2} + a^2 \left[\frac{\partial^2}{\partial \theta^2} + 2(2-\nu) \frac{\partial^4}{\partial \xi^2 \partial \theta^2} + 2 \frac{\partial^4}{\partial \theta^4} + \left(\frac{1-\nu}{2} \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2}\right) \nabla^4\right]$$

$$D_{12} = -\frac{\partial^2}{\partial \xi \, \partial \theta} \left\{\frac{1-\nu}{2} + a^2 \left[\nu(2-\nu) \frac{\partial^2}{\partial \xi^2} + \nu \frac{\partial^2}{\partial \theta^2} + \frac{1+\nu}{2} \nabla^4\right]\right\}$$

$$D_{13} = \frac{1-\nu}{2} \left(\nu \frac{\partial^3}{\partial \xi^3} - \frac{\partial^3}{\partial \xi \, \partial \theta^2}\right) + \frac{1+\nu}{2} a^2 \left[(2-\nu) \frac{\partial^5}{\partial \xi^3 \partial \theta^2} + \frac{\partial^5}{\partial \xi \, \partial \theta^4}\right]$$

$$D_{22} = \frac{1-\nu}{2} \left[2(1+\nu) \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2}\right] + a^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\nu}{2} \frac{\partial^2}{\partial \theta^2}\right) \nabla^4$$

$$D_{23} = \frac{1-\nu}{2} \left[(2+\nu) \frac{\partial^3}{\partial \xi^2 \partial \theta} - \frac{\partial^3}{\partial \theta^3}\right] - a^2 \left[(2-\nu) \frac{\partial^5}{\partial \xi^4 \partial \theta} + \right.$$

$$\left. + \frac{4 - 3\nu + \nu^2}{2} \frac{\partial^5}{\partial \xi^2 \partial \theta^3} + \frac{1-\nu}{2} \frac{\partial^5}{\partial \theta^5}\right], \qquad D_{33} = \frac{1-\nu}{2} \nabla^4$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2}\right) \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2}\right), \qquad a^2 = \frac{h^2}{12R^2} \qquad (1.8)$$

Here ν and h are the Poisson coefficient and shell thickness, respectively. The operator D corresponding to (1.8) is:

$$D = \nabla^4 \nabla^4 + 4\varkappa^4 \frac{\partial^4}{\partial \xi^4} + 2(4-\nu^2) \frac{\partial^6}{\partial \xi^4 \partial \theta^2} + 8 \frac{\partial^6}{\partial \xi^2 \partial \theta^4} + 2 \frac{\partial^6}{\partial \theta^6} + 4 \frac{\partial^4}{\partial \xi^2 \partial \theta^2} + \frac{\partial^4}{\partial \theta^4} \qquad (1.9)$$

$$4\varkappa^4 = \frac{1-\nu^2}{a^2}$$

The underlined terms in (1.8) and (1.9)(*) correspond to the theory of shallow cylindrical shells. The displacements due to the moments M_1 and M_2 are determined by the third member in (1.7) if Ψ_{3M} from (1.6) is substituted in place of Ψ_3.

Let ε_1, ε_2, ω, $\varkappa_1$, $\varkappa_2$ respectively, denote the longitudinal and circumferential strains, the shear strain of the middle surface, and the longitudinal and circumferential bending strain.

Taking account of (1.7) and (1.4), we obtain from geometric relationships

$$\varepsilon_1(\xi,\theta)=\frac{2(1+\nu)}{\pi^2 Eh}\int_l\sum_{j=1}^{3}q_j(\xi_0,\theta_0)\,\varepsilon_{1j}(\xi-\xi_0,\theta-\theta_0)\,dl \tag{1.10}$$

The strains $\varepsilon_2, \omega, R\varkappa_1, R\varkappa_2$ are of analogous form, and are represented in terms of the appropriate Green's functions ε_{2j}, ω_j, $\varkappa_{1j}$, $\varkappa_{2j}$. The stress resultants and moments are expressed in terms of (1.10) from physical considerations.

The described method is convenient in that the solution of the problem is reduced to determining the Green's function G from (1.5) independently of the form of the external forces. The strain and stress resultant components are determined by integration in Formulas of the type (1.10). The method is suitable for loadings acting along arbitrarily disposed lines, as well as for arbitrary surface loadings. It is conveniently used in problems where the stress resultant q_j is to be determined. In this case it is necessary to solve integral equations.

2. Taking account of (1.3), the particular solution of (1.5) is taken as

$$G(\xi,\theta)=-{}^1/_2G_0(\xi)-\sum_{n=1}^{\infty}G_n(\xi)\cos n\theta \tag{2.1}$$

Substituting (2.1) into (1.5) for $G_n(\xi)$, we obtain ordinary inhomogeneous differential equations whose solution is

$$G_1(\xi)\approx G_0(\xi)=\frac{\pi}{32\varkappa^4}\left[\frac{2}{3}|\xi|^3-\frac{e^{-\varkappa|\xi|}}{\varkappa^3}(\cos\varkappa\xi+\sin\varkappa|\xi|)\right] \tag{2.2}$$

$$G_n(\xi)=\frac{\pi}{4n^7\Delta}\sum_{j=1}^{2}\frac{e^{-q_jn|\xi|}}{p_jq_j(p_j^2+q_j^2)}(a_j\cos p_jn\xi+b_j\sin p_jn|\xi|) \tag{2.3}$$

where p_j and q_j are, respectively, the real and imaginary parts of the roots of the characteristic equation corresponding to the differential equation for $G_n(\xi)$, wherein the values of q_j are taken positive

$$a_j=p_j[A^2\mp4(p_1^2q_1^2-p_2^2q_2^2)\mp4Aq_j^2],\quad b_j=q_j[A^2\mp4(p_1^2q_1^2-p_2^2q_2^2)\pm4Ap_j^2] \tag{2.4}$$

$$\Delta=[A^2+4(p_1q_1+p_2q_2)^2][A^2+4(p_1q_1-p_2q_2)^2],\qquad A=p_1^2-q_1^2-p_2^2+q_2^2$$

The upper sign is taken in (2.4) for $j=1$, and the lower for $j=2$. It is assumed in obtaining $G_1(\xi)$ that $\varkappa^2\gg1$.

3. In general, the Green's functions $\varepsilon_{ij}, \omega_j, \varkappa_{ij}$ $(i=1, 2;\ j=1, 2, 3)$, in Formulas of type (1.10) increase without limit in the neighborhood of the point $\xi=\xi_0$, $\theta=\theta_0$. We obtain the principal value ε_{2j}°, ω_j°, $\varkappa_{ij}^\circ$ of these functions if we take G° from the solution of Eq.

$$\nabla^4\nabla^4G^\circ(\xi,\theta)+\delta(\xi,\theta)=0 \tag{3.1}$$

in place of the Green's function from (1.5).

This solution is

$$G^\circ(\xi,\theta)=-\frac{\pi}{96}\sum_{n=1}^{\infty}\frac{e^{-n|\xi|}}{n^7}[(n|\xi|)^3+6(n|\xi|)^2+15\ (n|\xi|+1)]\cos n\theta-\frac{1}{4}\frac{\pi}{7!}|\xi|^7$$

*) Editor's note: No terms have been underlined in the Russian original.

Operating on G° exactly as on G in obtaining the Green's function in (1.10), and hence retaining only the highest derivatives in the differential operators for ε_{ij}, ω_j, $\varkappa_{ij}$ (because only such operators may yield unbounded components for ε_{ij}, ω_j, $\varkappa_{ij}$), we obtain

$$\varepsilon_{11}^{\circ} = -\frac{\pi}{4}\left[(1-\nu)\varphi_1 + \frac{1+\nu}{2}\varphi_2\right], \quad \varepsilon_{12}^{\circ} = -\frac{\pi(1+\nu)}{8}(\psi_2 - \psi_1)$$

$$\varepsilon_{21}^{\circ} = \frac{\pi(1+\nu)}{8}\varphi_2, \quad \varepsilon_{22}^{\circ} = -\frac{\pi}{4}\left[\frac{3-\nu}{2}\psi_1 - \frac{1+\nu}{2}\psi_2\right] \tag{3.2}$$

$$\omega_1^{\circ} = -\frac{\pi}{4}[(1-\nu)\psi_1 + (1+\nu)\psi_2], \quad \omega_2^{\circ} = -\frac{\pi}{4}[2\varphi_1 - (1+\nu)\varphi_2]$$

$$\varkappa_{13}^{\circ} = \varkappa_{23}^{\circ} = \frac{\pi(1+\nu)}{8a^2}\left[\frac{1}{2}|\xi| - \frac{1}{2}\ln(\operatorname{ch}\xi - \cos\theta)\right]$$

where

$$\varphi_1 = \left[\sum_{n=1}^{\infty} e^{-n|\xi|}\cos n\theta + \frac{1}{2}\right]\operatorname{sig} n\xi = \frac{1}{2}\frac{\operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\theta}$$

$$\varphi_2 = \sum_{n=1}^{\infty} n|\xi| e^{-n|\xi|}\cos n\theta \operatorname{sig} n\xi = -\frac{1}{2}\xi\frac{1-\operatorname{ch}\xi\cos\theta}{(\operatorname{ch}\xi - \cos\theta)^2} \tag{3.3}$$

$$\psi_1 = \sum_{n=1}^{\infty} e^{-n|\xi|}\sin n\theta = \frac{1}{2}\frac{\sin\theta}{\operatorname{ch}\xi - \cos\theta}$$

$$\psi_2 = \sum_{n=1}^{\infty} n|\xi| e^{-n|\xi|}\sin n\theta = \frac{1}{2}\xi\frac{\operatorname{sh}\xi\sin\theta}{(\operatorname{ch}\xi - \cos\theta)^2}$$

Let us note that formulas of type (1.10) yield a closed solution for an infinite plate loaded by forces q_j along an infinite number of segments arranged at $2\pi R$ intervals in the direction of the $y = R\theta$ axis, if the functions (3.2) are substituted in place of ε_{ij}, ω_j, $\varkappa_{ij}$. Such a plate is obtained if the cylindrical shell is represented as an infinite sheeted surface (roll of material) each of whose branches is loaded exactly as is the considered shell, and then such a surface with the loading is unrolled on a plane. If the plate is not bent ($q_3 = 0$), then we have the following values of the stress resultant forces at infinity: $T_1 = \sigma_x h$ and $T_2 = \sigma_y h$ and for the shear

$$T_1 = -\frac{1}{2}\left[\frac{1}{2\pi R}\int_l q_1 R\, dl\right]\operatorname{sign}\xi, \quad T_2 = -\nu T_1, \quad S = -\frac{1}{2}\frac{1}{2\pi R}\int_l q_2 R\, dl \tag{3.4}$$

In bending the obtained results may be considered as a particular integral for an infinite strip with edges parallel to the y-axis.

The principal values of the Green's functions (3.2) permit asymptotic formulas to be obtained for strains and stresses increasing without limit to the neighborhood of the ends of the loading segments. Asymptotic formulas are presented below for the case when the loading is along a segment of the generator or arc of a circle. Two cases are considered separately:

a) When arbitrarily distributed stress resultants q_1 and q_2 are bounded on the segment $a < \xi < b$ of the generator. In this case the asymptotic formulas are

$$T_1 \approx \mp\frac{3+\nu}{4\pi}q_1(c)\ln\rho_c, \quad T_2 \approx \pm\frac{1-\nu}{4\pi}q_1(c)\ln\rho_c, \quad S \approx \mp\frac{1-\nu}{4\pi}q_2(c)\ln\rho_c \tag{3.5}$$

where T_1, T_2, S are the longitudinal and circumferential tensile forces and the shear force of the middle surface, respectively; ρ_c is the distance from the point c, coincident with either a or b, to the point at which the forces are sought. The upper sign is taken for the neighborhood of the point a, and the lower for b. The forces T_1 and T_2 due to the loading

q_2, and also the force S due to q_1, will be bounded. If the functions q_1 and q_2 vanish at the endpoints of the segment, then all the forces will be bounded;

b) The forces q_1 and q_2 in the neighborhoods of the endpoints a and b are, respectively

$$q_j = q_j^\circ / \sqrt{\xi - a}, \quad q_j = q_j^\circ / \sqrt{b - \xi} \quad (j = 1, 2) \qquad (q_j^\circ = \text{const})$$

We obtain such a nature of the singularities, for example, for tangential forces q_1 transmitted to the shell from stringers loaded at the ends by similar forces.

In this case

$$T_1 \approx \pm \frac{q_1^\circ}{4\sqrt{\rho_c}} \left[3 + \nu + (1 + \nu) \cos \frac{\alpha}{2} \cos \frac{3\alpha}{2}\right] \sin \frac{\alpha}{2}$$

$$T_2 \approx \mp \frac{q_1^\circ}{4\sqrt{\rho_c}} \left[1 - \nu + (1 + \nu) \cos \frac{\alpha}{2} \cos \frac{3\alpha}{2}\right] \sin \frac{\alpha}{2} \tag{3.6}$$

$$S_1 \approx \mp \frac{q_1^\circ}{2\sqrt{\rho_c}} \left(1 - \frac{1+\nu}{2} \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2}\right) \cos \frac{\alpha}{2}$$

$$T_1 \approx \frac{q_2^\circ}{4\sqrt{\rho_c}} \left(-\nu + \frac{1+\nu}{2} \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2}\right) \cos \frac{\alpha}{2}$$

$$T_2 \approx -\frac{q_2^\circ}{4\sqrt{\rho_c}} \left(1 + \frac{1+\nu}{2} \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2}\right) \cos \frac{\alpha}{2}$$

$$S_1 \approx \pm \frac{q}{4\sqrt{\rho_c}} \left[1 - \nu - (1 + \nu) \cos \frac{\alpha}{2} \cos \frac{3\alpha}{2}\right] \sin \frac{\alpha}{2}$$

where α is the angle between the segment (a, b) and a vector connecting the point a or b to the point z where the appropriate force is determined. This angle is measured from the segment (a, b) counter-clockwise if z belongs to the neighborhood of the point a and clockwise if z belongs to the neighborhood of the point b. As before, the upper sign is taken for the point a in formulas with the double sign.

Formulas (3.5) and (3.6) remain valid even for forces acting along the segment of an arc if T_1, T_2, S, q_1, q_2 are replaced by T_2, T_1, $-S$, q_2, $-q_1$, respectively.

The behavior of the moments in the neighborhood of the points a and b of the shell changes when one version of the theory is replaced by another. When using different versions of the "exact" theory, the nature of the singularities will be the same as for the forces, however, the asymptotic formulas will differ for the different modifications. If we start from shallow shell theory, the moments due to the forces q_1 and q_2 will be bounded. Under the effect of radial forces q_3 the membrane stresses will be bounded, there are no moments.

Let us note that the asymptotic Formulas (3.5) and (3.6) are identical to the corresponding formulas for plates. In the particular case when q_1 and q_2 are constants, Formulas (3.5) agree with the Formulas in [6].

4. In investigating the state of stress of cylindrical shells reinforced by stringers or frames which take on the external forces, the problem of determining the forces in such kind of stiffness elements, and the forces being transmitted from the stiffness elements to the shell has to be considered. In this case the loadings q_j in formulas of type (1.10) are sought. From the conditions of the connection between the stringers, frames and the shell it is possible to obtain integral equations to find the forces q_j. These equations are often singular because of the singularity of the Green's functions ε_{ij}, which are defined in this case on the line of action of the loading on the generator or arc of a circle. In similar problems it is especially important to isolate the principal value of the Green's function since this affords an opportunity to utilize the well developed theory of singular integral equations. The authors used such a procedure in examining loading transmission from stringers to a shell. Let us simplify the form of the principal values of the Green's functions so that they could conveniently be used in connection problems.

It can be shown that the asymptotic equality

$$\sum_{n=1}^{\infty} (n\,|\,\xi\,|)^m\, l^{-n|\xi|} \approx m! \sum_{n=1}^{\infty} l^{-n|\xi|} = \frac{m!}{2}\left(\operatorname{cth}\frac{|\,\xi\,|}{2} - 1\right)$$

is valid in the neighborhood of the point $\xi = 0$.

Taking account of this equality, the following can be taken on the generator $\theta = 0$ in place of the asymptotic formulas (3.2):

$$\varepsilon_{11}{}^{\circ}(\xi) = -\frac{\pi(3-\nu)}{16}\operatorname{cth}\frac{\xi}{2}, \qquad \varepsilon_{21}{}^{\circ} = \frac{\pi(1+\nu)}{16}\operatorname{cth}\frac{\xi}{2} \tag{3.7}$$

$$\omega_2{}^{\circ}(\xi) = -\frac{\pi(1-\nu)}{8}\operatorname{cth}\frac{\xi}{2}, \qquad \varkappa_{13}{}^{\circ}(\xi) = \varkappa_{23}{}^{\circ}(\xi) = -\frac{\pi(1-\nu)}{8a^2}\ln\left(2\operatorname{sh}\frac{|\,\xi\,|}{2}\right)$$

The functions $\varepsilon_{12}{}^{\circ}(\xi)$, $\omega_1{}^{\circ}(\xi)$, $\varepsilon_{22}{}^{\circ}(\xi)$ equal zero at $\theta = 0$.

From (3.2) we find on the arcs of the circle $\xi = 0$

$$\varepsilon_{12}{}^{\circ}(\theta) = \frac{\pi(1+\nu)}{16}\operatorname{ctg}\frac{\theta}{2}, \qquad \varepsilon_{22}{}^{\circ} = -\frac{\pi(3-\nu)}{16}\operatorname{ctg}\frac{\theta}{2} \tag{3.8}$$

$$\omega_1{}^{\circ}(\theta) = -\frac{\pi(1-\nu)}{8}\operatorname{cth}\frac{\theta}{2}, \qquad \varkappa_{13}{}^{\circ}(\theta) = \varkappa_{23}{}^{\circ}(\theta) = -\frac{\pi(1-\nu)}{8a^2}\ln\left(2\sin\frac{\theta}{2}\right)$$

The functions $\varepsilon_{11}{}^{\circ}$, $\varepsilon_{21}{}^{\circ}$, $\omega_2{}^{\circ}$ equal zero at $\xi = 0$.

Integral equations with kernels of the form (3.7) may be solved either directly or by reduction to equations with a Cauchy kernel. The theory of equations with the kernels (3.8), namely, equations with Hilbert kernels, has also been well worked out.

BIBLIOGRAPHY

1. Odquist, F.K.G., Action of forces and moments symmetrically distributed along a generatrix of a thin cylindrical shell. J. Appl. Mech., Vol. 13, No. 2, 1946.
2. Shoessow, G.L. and Kooistra, L.F., Stresses in a cylindrical shell due to nozzle and pipe connections. Trans. ASME, Vol. 67, pt. A-107, 1945.
3. Hoff, N.J., Kempner, J., and Pohle, F.V., Line load applied along generators of thin-walled circular cylindrical shells of finite length. Quart. Appl. Math., Vol. 11, No. 4, 1954.
4. Gol'denveizer, A.L., Theory of Elastic Thin Shells, Moscow, Gostekhizdat, 1953,
5. Juan, S.W., Thin cylindrical shells subjected to concentrated loads. Quart. Appl. Math., Vol. 4, No. 1, 1946.
6. Darevskii, V.M., On the theory of cylindrical shells. PMM, Vol. 15, No. 5, 1951.

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